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A NOTE ON INTEGRABILITY CONDITIONS AND TOTALLY GEODESIC FOLIATIONS OF DISTRIBUTIONS ON A SEMI-SLANT LIGHTLIKE SUBMANIFOLD

Muqeem Ahmad and Mobin Ahmad

Department of Mathematics and Statistics,
Faculty of Science, Integral University,
Lucknow - 226026, Uttar Pradesh, INDIA

E-mail : muqeem.jmi@gmail.com, mobinahmad68@gmail.com

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Abstract: In the present paper, we consider a golden semi-Riemannian manifold and find its semi-slant lightlike submanifolds. We examine the integrability of the distributions D_1, D_2 and $Rad\mathtt{TJ}$ defined on semi-slant lightlike submanifolds. Further, we investigate the conditions of the foliations of the distributions when they become totally geodesic. The goal of this study is to determine the semi-slant lightlike submanifolds of a golden semi-Riemannian manifold. The distributions D_1, D_2, and $Rad\mathtt{TJ}$ defined on semi-slant lightlike submanifolds are examined for integrability. Additionally, we look at the circumstances surrounding the distributions' foliations when they reach complete geodesic.

Keywords and Phrases: Degenerate metric, indefinite metrics, semi-slant lightlike submanifold, golden structure, golden semi-Riemannian manifold.

2020 Mathematics Subject Classification: 53C15, 53C25, 53C40, 53C50.

1. Introduction

In [8], Duggal and Bejancu introduced the notion of lightlike submanifolds of semi-Riemannian manifolds. Since then, many geometers studied geometry of lightlike submanifolds (see [3], [9], [25], [20], [10], [21]). A submanifold $\mathtt{J}$ of a semi-Riemannian manifold $\overline{\mathtt{J}}$ is said to be a lightlike submanifold if the induced metric g on $\mathtt{J}$ is degenerate, i.e, $g(\mathtt{Q}, \mathtt{W}) = 0$ for any non-zero $\mathtt{Q} \in \Gamma(\mathtt{TJ})$ and $\forall\ \mathtt{W} \in \Gamma(\mathtt{TJ})$.

Lightlike geometry has its applications in general theory of relativity, particularly in black hole theory. Infact, lightlike hypersurfaces are examples of physical model of Killing horizons in general relativity. The surface of black hole is described in terms of Killing horizon. In ([5], [6]), B. Y. Chen introduced slant submanifolds of an almost Hermitian manifold in which angle Ψ between $\bar{\mathtt{X}}\mathtt{Q}$ and the tangent space is constant for any tangent vector field $\mathtt{Q}$. Slant submanifolds are the generalization of the complex submanifolds ($\Psi = 0$) and totally real submanifolds ($\Psi = \frac{\pi}{2}$). A. Lotta [18] investigated the concept of slant submanifolds in contact geometry. J. L. Cabrerizo et. al. analyzed slant submanifolds of a Sasakian manifold [4] and B. Sahin studied slant submanifolds in indefinite Hermitian manifold and almost product manifolds ([24], [23]). The geometry of semi-slant submanifolds of Kaehler manifolds was studied by [19]. H. Li and X. Liu analyzed semi-slant submanifolds of a locally product manifold in [17].

On the other hand, the notion of golden structure on a Riemannian manifold was introduced for the first time by C. E. Hretcanu and M. Crasmareanu in [7]. Moreover, the authors investigated the properties of golden structure in ([13], [14]). M. A. Qayyoom and M. Ahmad studied hypersurfaces of golden Riemannian manifolds in [22]. Also, they studied submanifolds of locally metallic Riemannian manifolds in [2]. A. Gezer et. al. investigated the integrability of golden Riemannian structure in [12]. N. Poyraz and E. Yasar studied lightlike submanifolds of golden semi-Riemannian manifolds in [20]. F. E. Erdogan et. al. investigated transversal lightlike submanifolds of golden semi-Riemannian manifolds in [10]. S. S. Shukla and A. Yadav studied semi-slant lightlike submanifolds of indefinite Kaehler manifold in [26]. Moreover, the authors also investigated semi-slant lightlike submanifolds and screen semi-slant lightlike submanifolds of indefinite Sasakian manifolds in ([27], [28]). B. E. Acet studied screen pseudo slant lightlike submanifolds of golden semi-Riemannian manifolds in [1]. Semi-slant lightlike submanifolds of golden semi-Riemannian manifolds were defined and studied by S. Kumar and A. Yadav in [16]. Johnson and Whitt studied foliations that has great importance in differential geometry, they considered each leaf of the foliation to be a totally geodesic submanifold of the ambient space [15].

The study of lightlike submanifolds has drawn considerable interest in differential geometry due to its importance in theoretical physics, particularly in the context of spacetime models and general relativity [11]. Moreover, the investigation of semi-slant submanifolds, pioneered by Li and Liu [17], has provided insights into geometric structures beyond the classical Riemannian setting. Despite substantial study on lightlike and slant submanifolds, there is still a gap in understanding the relationship between semi-slant geometry and the golden ratio in the setting of semi-Riemannian manifolds. While significant progress has been made in understanding the geometric properties of lightlike submanifolds and semi-Riemannian manifolds individually, the exploration of their interaction with the golden ratio remains relatively unexplored. Previous works have primarily focused on classical geometries or specific types of submanifolds, leaving a notable research vacuum in the study of semi-slant lightlike submanifolds within the context of golden semi-Riemannian manifolds. This article seeks to fill that gap by providing a detailed analysis of semi-slant lightlike submanifolds in the context of golden semi-Riemannian manifolds.

In this paper, we study the notion of semi-slant lightlike submanifold of a golden semi-Riemannian manifold. We give two examples of semi-slant lightlike submanifold and a characterization theorem of it. We obtain integrability conditions of the distributions and give necessary and sufficient conditions for foliations determined by the distributions to be totally geodesic.

2. Preliminaries

Let $(\overline{\mathtt{J}}, g)$ be a semi-Riemannian manifold. A golden structure on $(\overline{\mathtt{J}}, g)$ is a non-null tensor $\overline{\mathtt{X}}$ of type (1,1) which satisfies the equation

$$\overline{\mathtt{X}}^2 = \overline{\mathtt{X}} + I, \tag{2.1}$$

where I is the identity transformation. We say that the metric g is $\overline{\mathtt{X}}$-compatible if

$$g(\overline{\mathtt{X}}\mathtt{Q}, \mathtt{W}) = g(\mathtt{Q}, \overline{\mathtt{X}}\mathtt{W}) \tag{2.2}$$

for all $\mathtt{Q}, \mathtt{W}$ vector fields on $\overline{\mathtt{J}}$. If we substitute $\overline{\mathtt{X}}\mathtt{Q}$ into $\mathtt{Q}$ in (2.2), then we have

$$g(\overline{\mathtt{X}}\mathtt{Q}, \overline{\mathtt{X}}\mathtt{W}) = g(\overline{\mathtt{X}}\mathtt{Q}, \mathtt{W}) + g(\mathtt{Q}, \mathtt{W}). \tag{2.3}$$

The semi-Riemannian metric is called $\overline{\mathtt{X}}$-compatible and $(\overline{\mathtt{J}}, \overline{\mathtt{X}}, g)$ is called a golden semi-Riemannian manifold. Also, if

$$\overline{\nabla}_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W} = \overline{\mathtt{X}}\ \overline{\nabla}_{\mathtt{Q}}\mathtt{W} \tag{2.4}$$

for all $\mathtt{Q}, \mathtt{W} \in \Gamma(\mathtt{T}\overline{\mathtt{J}})$, where $\overline{\nabla}$ is the Levi-Civita connection with respect to g, then $(\overline{\mathtt{J}}, \overline{\mathtt{X}}, g)$ is called locally golden semi-Riemannian manifold.
A submanifold $(\mathtt{J}^m, g)$ immersed in a semi-Riemannian manifold $(\overline{\mathtt{J}}^{m+n}, \overline{g})$ is called a lightlike submanifold if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $Rad(\mathtt{TJ})$ is of rank r,where $1 \leq r \leq m$. Let $\mathtt{S}(\mathtt{TJ})$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(\mathtt{TJ})$ in $\mathtt{TJ}$, i.e., $\mathtt{TJ} = Rad(\mathtt{TJ}) \perp \mathtt{S}(\mathtt{TJ})$.

Consider a screen transversal vector bundle $\mathtt{S}(\mathtt{TJ}^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(\mathtt{TJ})$ in $\mathtt{TJ}^{\perp}$. Since, for any local basis $\{\mathtt{U}_i\}$ of $Rad(\mathtt{TJ})$, there exists a local null frame $\{\mathtt{Y}_i\}$ of sections with values in the orthogonal complement of $\mathtt{S}(\mathtt{TJ}^{\perp})$ in $[\mathtt{S}(\mathtt{TJ})]^{\perp}$ such that $\overline{g}(\mathtt{U}_i, \mathtt{Y}_j) = \delta_{ij}$, it follows that there exists a lightlike transversal vector bundle $ltr(\mathtt{TJ})$ locally spanned by $\{\mathtt{Y}_i\}$. Let $tr(\mathtt{TJ})$ be complementary(but not orthogonal) vector bundle to $\mathtt{TJ}$ in $\mathtt{T\overline{J}}|_{\mathtt{J}}$. Then,

$$tr(\mathtt{TJ}) = ltr(\mathtt{TJ}) \perp \mathtt{S}(\mathtt{TJ}^{\perp}),$$

$$\mathtt{T\overline{J}}|_{\mathtt{J}} = \mathtt{S}(\mathtt{TJ}) \perp [Rad(\mathtt{TJ}) \bigoplus ltr(\mathtt{TJ})] \perp \mathtt{S}(\mathtt{TJ}^{\perp}).$$

A submanifold $(\mathtt{J}, g, \mathtt{S}(\mathtt{TJ}), \mathtt{S}(\mathtt{TJ}^{\perp}))$ of $\overline{\mathtt{J}}$ is said to be
(i) r-lightlike if $r < min\{m, n\}$;
(ii) Coisotropic if $r = n < m, \mathtt{S}(\mathtt{TJ}^{\perp}) = \{0\}$;
(iii) Isotropic if $r = m < n, \mathtt{S}(\mathtt{TJ}) = \{0\}$;
(iv) Totally lightlike if $r = m = n, \mathtt{S}(\mathtt{TJ}) = \{0\} = \mathtt{S}(\mathtt{TJ}^{\perp})$.
Let $\overline{\nabla}$, ∇ and ∇^t denote the linear connections on $\overline{\mathtt{J}}$, $\mathtt{J}$ and vector bundle $tr(\mathtt{TJ})$, respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_{\mathtt{Q}}\mathtt{W} = \nabla_{\mathtt{Q}}\mathtt{W} + h(\mathtt{Q}, \mathtt{W}), \quad \forall \;\; \mathtt{Q}, \mathtt{W} \in \Gamma(\mathtt{TJ}), \tag{2.5}$$

$$\overline{\nabla}_{\mathtt{Q}}\mathtt{Z} = -A_{\mathtt{Z}}\mathtt{Q} + \nabla^t_{\mathtt{Q}}\mathtt{Z}, \quad \forall \;\; \mathtt{Q} \in \Gamma(\mathtt{TJ}), \mathtt{Z} \in \Gamma tr(\mathtt{TJ}),$$

where $\{\nabla_{\mathtt{Q}}\mathtt{W}, A_{\mathtt{Z}}\mathtt{Q}\}$ and $\{h(\mathtt{Q}, \mathtt{W}), \nabla^t_{\mathtt{Q}}\mathtt{Z}\}$ belong to $\Gamma(\mathtt{TJ})$ and $\Gamma(ltr(\mathtt{TJ}))$, respectively. ∇ and $\nabla^t_{\mathtt{Q}}$ are linear connections on $\mathtt{J}$ and on the vector bundle $ltr(\mathtt{TJ})$, respectively. The second fundamental form h is a symmetric $F(\mathtt{J})$-bilinear form on $\Gamma(\mathtt{TJ})$ with values in $\Gamma(tr(\mathtt{TJ}))$ and the shape operator $A_{\mathtt{Z}}$ is a linear endomorphism of $\Gamma(\mathtt{TJ})$. Then we have

$$\overline{\nabla}_{\mathtt{Q}}\mathtt{W} = \nabla_{\mathtt{Q}}\mathtt{W} + h^l(\mathtt{Q}, \mathtt{W}) + h^s(\mathtt{Q}, \mathtt{W}), \tag{2.6}$$

$$\overline{\nabla}_{\mathtt{Q}}\mathtt{Y} = -A_{\mathtt{Y}}\mathtt{Q} + \nabla^l_{\mathtt{Q}}(\mathtt{Y}) + D^s(\mathtt{Q}, \mathtt{Y}), \tag{2.7}$$

$$\overline{\nabla}_{\mathtt{Q}}\mathtt{M} = -A_{\mathtt{M}}\mathtt{Q} + \nabla^s_{\mathtt{Q}}(\mathtt{M}) + D^l(\mathtt{Q}, \mathtt{M}), \quad \forall \;\; \mathtt{Q}, \mathtt{W} \in \Gamma(\mathtt{TJ}), \mathtt{Y} \in \Gamma(ltr(\mathtt{TJ})) \tag{2.8}$$

and $\mathtt{M} \in \Gamma(\mathtt{S}(\mathtt{TJ}^{\perp}))$. We denote the projection of $\mathtt{TJ}$ on $\mathtt{S}(\mathtt{TJ})$ by $\overline{P}$. Then, by using (2.5), (2.6)-(2.8) and $\overline{\nabla}$ being a metric connection we have

$$\overline{g}(h^s(\mathtt{Q}, \mathtt{W}), \mathtt{M}) + \overline{g}(\mathtt{W}, D^l(\mathtt{Q}, \mathtt{M})) = g(A_{\mathtt{M}}\mathtt{Q}, \mathtt{W}),$$

$$\overline{g}(D^s(\mathtt{Q}, \mathtt{Y}), \mathtt{M}) = \overline{g}(\mathtt{Y}, A_{\mathtt{M}}\mathtt{Q}).$$

We set

$$\nabla_{\mathtt{Q}}\overline{P}\mathtt{W} = \nabla^*_{\mathtt{Q}}\overline{P}\mathtt{W} + h^*(\mathtt{Q}, \overline{P}\mathtt{W}),$$

$$\nabla_{\mathtt{Q}}\mathtt{U} = -A^*_{\mathtt{U}}\mathtt{Q} + \nabla^{*t}_{\mathtt{Q}}\mathtt{U}$$

for $\mathtt{Q}, \mathtt{W} \in \Gamma(\mathtt{TJ})$ and $\mathtt{U} \in \Gamma(Rad\mathtt{TJ})$. By using above equations, we obtain

$$\overline{g}(h^l(\mathtt{Q}, \overline{P}\mathtt{W}), \mathtt{U}) = g(A^*_{\mathtt{U}}\mathtt{Q}, \overline{P}\mathtt{W}),$$

$$\overline{g}(h^*(\mathtt{Q}, \overline{P}\mathtt{W}), \mathtt{Y}) = g(A_{\mathtt{Y}}\mathtt{Q}, \overline{P}\mathtt{W}),$$

$$\overline{g}(h^l(\mathtt{Q}, \mathtt{U}), \mathtt{U}) = 0, A^*_{\mathtt{U}}\mathtt{U} = 0.$$

In general, the induced connection ∇ on $\mathtt{J}$ is not metric connection. Since $\overline{\nabla}$ is a metric connection, by using (2.3) we get

$$(\nabla_{\mathtt{Q}}g)(\mathtt{W}, \mathtt{G}) = \overline{g}(h^l(\mathtt{Q}, \mathtt{W}), \mathtt{G}) + \overline{g}(h^l(\mathtt{Q}, \mathtt{G}), \mathtt{W}).$$

However, it is important to note that ∇^* is a metric connection on $\mathtt{S}(\mathtt{TJ})$. From now on, we briefly denote $(\mathtt{J}, g, \mathtt{S}(\mathtt{TJ}), \mathtt{S}(\mathtt{TJ}^{\perp}))$ by $\mathtt{J}$ in this paper.
To define semi-slant lightlike submanifolds we need the following lemma:

Lemma 2.1. [16] *Let $\mathtt{J}$ be a lightlike submanifold of index q of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$ whose index is $2q$. Consider a screen distribution $\mathtt{S}(\mathtt{TJ})$ in such a way that $\overline{\mathtt{X}}Rad\mathtt{TJ} \subset \mathtt{S}(\mathtt{TJ})$ and $\overline{\mathtt{X}}ltr(\mathtt{TJ}) \subset \mathtt{S}(\mathtt{TJ})$. Then, $\overline{\mathtt{X}}Rad\mathtt{TJ} \bigcap \overline{\mathtt{X}}ltr(\mathtt{TJ}) = \{0\}$ and the distribution which is complementary to $\overline{\mathtt{X}}Rad\mathtt{TJ} \bigoplus \overline{\mathtt{X}}ltr(\mathtt{TJ})$ in $\mathtt{S}(\mathtt{TJ})$ is Riemannian.*

Definition 2.2. *Let $\mathtt{J}$ be a lightlike submanifold of index q of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$ whose index is $2q$ in such a way that $2q < dim(\mathtt{J})$. Then, we say that $\mathtt{J}$ is a semi-slant lightlike submanifold of $\overline{\mathtt{J}}$ if the following conditions hold:*
(i) The distribution $\overline{\mathtt{X}}Rad\mathtt{TJ}$ on $\mathtt{J}$ is such that $Rad\mathtt{TJ} \bigcap \overline{\mathtt{X}}Rad\mathtt{TJ} = \{0\}$.
(ii) The distributions D_1 and D_2 on $\mathtt{J}$ are non-degenerate orthogonal in such a way that

$$\mathtt{S}(\mathtt{TJ}) = (\overline{\mathtt{X}}Rad\mathtt{TJ} \bigoplus \overline{\mathtt{X}}ltr(\mathtt{TJ})) \bigoplus_{orth} D_1 \bigoplus_{orth} D_2;$$

(iii) There is an invariant distribution D_1 , i.e. $\overline{\mathtt{X}}D_1 = D_1$;
(iv) There is a slant distribution D_2 having slant angle $\Psi(\neq 0)$, i.e. for each $\mathtt{Q} \in \mathtt{J}$ and each non-zero vector $\mathtt{Q} \in (D_2)_{\mathtt{Q}}$, the angle Ψ between $\overline{\mathtt{X}}\mathtt{Q}$ and the vector space $(D_2)_{\mathtt{Q}}$ is a non-zero constant, which does not depend on the choice of $\mathtt{Q} \in \mathtt{J}$ and $\mathtt{Q} \in (D_2)_{\mathtt{Q}}$.

The angle Ψ which is constant is called the slant angle of distribution D_2. A semi-slant lightlike submanifold is proper if $D_1 \neq \{0\}, D_2 \neq \{0\}$ and $\Psi \neq \frac{\pi}{2}$.

The definition stated above gives the decomposition as follows

$$\mathtt{TJ} = Rad\mathtt{TJ} \bigoplus_{orth} (\overline{\mathtt{X}} Rad\mathtt{TJ} \bigoplus \overline{\mathtt{X}} ltr(\mathtt{TJ})) \bigoplus_{orth} D_1 \bigoplus_{orth} D_2.$$

Particularly, we have
(i) If $D_1 = 0$, then $\mathtt{J}$ is a slant lightlike submanifold;
(ii) If $D_1 \neq 0$ and $\Psi = \frac{\pi}{2}$, then $\mathtt{J}$ is CR-lightlike submanifold.

3. Semi-slant lightlike submanifolds

Example 3.1. Let a semi-Euclidean space $(R_2^{12}, \overline{g})$ having signature $(-,-,+,+,+,+,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial \mathtt{q}_1, \partial \mathtt{q}_2, \partial \mathtt{q}_3, \partial \mathtt{q}_4, \partial \mathtt{q}_5, \partial \mathtt{q}_6, \partial \mathtt{q}_7, \partial \mathtt{q}_8, \partial \mathtt{q}_9, \partial \mathtt{q}_{10}, \partial \mathtt{q}_{11}, \partial \mathtt{q}_{12}\}$ with the golden structure $\overline{\mathtt{X}}$ defined as

$$\begin{aligned}\overline{\mathtt{X}}(\mathtt{q}_1, \mathtt{q}_2, \mathtt{q}_3, \mathtt{q}_4, \mathtt{q}_5, \mathtt{q}_6, \mathtt{q}_7, \mathtt{q}_8, \mathtt{q}_9, \mathtt{q}_{10}, \mathtt{q}_{11}, \mathtt{q}_{12}) = (&\Psi \mathtt{q}_1, \overline{\Psi} \mathtt{q}_2, \Psi \mathtt{q}_3, \overline{\Psi} \mathtt{q}_4, \overline{\Psi} \mathtt{q}_5, \overline{\Psi} \mathtt{q}_6, \\ &\overline{\Psi} \mathtt{q}_7, \overline{\Psi} \mathtt{q}_8, \overline{\Psi} \mathtt{q}_9, \overline{\Psi} \mathtt{q}_{10}, \Psi \mathtt{q}_{11}, \Psi \mathtt{q}_{12}).\end{aligned}$$

Let a 7-dimensional submanifold $\mathtt{J}$ of $(R_2^{12}, \overline{g})$ given as

$$\begin{aligned}&\mathtt{q}_1 = \mathtt{w}_1 + \Psi \mathtt{w}_2 - \Psi \mathtt{w}_3, \mathtt{q}_2 = \Psi \mathtt{w}_1 - \mathtt{w}_2 + \mathtt{w}_3, \mathtt{q}_3 = \mathtt{w}_1 + \Psi \mathtt{w}_2 + \Psi \mathtt{w}_3, \mathtt{q}_4 = -\Psi \mathtt{w}_1 + \mathtt{w}_2 + \mathtt{w}_3, \\ &\mathtt{q}_5 = \overline{\Psi} \mathtt{w}_4, \mathtt{q}_6 = \overline{\Psi} \mathtt{w}_5, \mathtt{q}_7 = \Psi \mathtt{w}_4, \mathtt{q}_8 = \Psi \mathtt{w}_5, \mathtt{q}_9 = \overline{\Psi} \mathtt{w}_6, \mathtt{q}_{10} = \overline{\Psi} \mathtt{w}_7, \mathtt{q}_{11} = \Psi \mathtt{w}_6, \mathtt{q}_{12} = \Psi \mathtt{w}_7.\end{aligned}$$

Then, $\mathtt{TJ}$ is spanned by $\mathtt{G}_1, \mathtt{G}_2, \mathtt{G}_3, \mathtt{G}_4, \mathtt{G}_5, \mathtt{G}_6, \mathtt{G}_7$, where

$$\begin{aligned}&\mathtt{G}_1 = \partial \mathtt{q}_1 + \Psi \partial \mathtt{q}_2 + \partial \mathtt{q}_3 - \Psi \partial \mathtt{q}_4, \mathtt{G}_2 = \Psi \partial \mathtt{q}_1 - \partial \mathtt{q}_2 + \Psi \partial \mathtt{q}_3 + \partial \mathtt{q}_4, \\ &\quad \mathtt{G}_3 = -\Psi \partial \mathtt{q}_1 + \partial \mathtt{q}_2 + \Psi \partial \mathtt{q}_3 + \partial \mathtt{q}_4, \mathtt{G}_4 = \overline{\Psi} \partial \mathtt{q}_5 + \Psi \partial \mathtt{q}_7, \mathtt{G}_5 = \overline{\Psi} \partial \mathtt{q}_6 + \Psi \partial \mathtt{q}_8, \\ &\qquad \mathtt{G}_6 = \overline{\Psi} \partial \mathtt{q}_9 + \Psi \partial \mathtt{q}_{11}, \mathtt{G}_7 = \overline{\Psi} \partial \mathtt{q}_{10} + \Psi \partial \mathtt{q}_{12}.\end{aligned}$$

This implies that $Rad\mathtt{TJ} = span\{\mathtt{G}_1\}$ and $\mathtt{S}(\mathtt{TJ}) = span\{\mathtt{G}_2, \mathtt{G}_3, \mathtt{G}_4, \mathtt{G}_5, \mathtt{G}_6, \mathtt{G}_7\}$.

Now, $ltr(\mathtt{TJ}) = span\{\mathtt{Y}\}$ where $\mathtt{Y}$ is given by

$$\mathtt{Y} = \frac{1}{2(2+\Psi)}\{-\partial \mathtt{q}_1 - \Psi \partial \mathtt{q}_2 + \partial \mathtt{q}_3 - \Psi \partial \mathtt{q}_4\}.$$

and $\mathtt{S}(\mathtt{TJ}^\perp) = span\{\mathtt{M}_1, \mathtt{M}_2, \mathtt{M}_3, \mathtt{M}_4\}$ where

$$\mathtt{M}_1 = \Psi \partial \mathtt{q}_5 - \overline{\Psi} \partial \mathtt{q}_7, \mathtt{M}_2 = \Psi \partial \mathtt{q}_6 - \overline{\Psi} \partial \mathtt{q}_8, \mathtt{M}_3 = \Psi \partial \mathtt{q}_9 - \overline{\Psi} \partial \mathtt{q}_{11}, \mathtt{M}_4 = \Psi \partial \mathtt{q}_{10} - \overline{\Psi} \partial \mathtt{q}_{12}.$$

Now, $\overline{\mathtt{X}}\mathtt{G}_1 = \mathtt{G}_2, \overline{\mathtt{X}}\mathtt{Y} = \mathtt{G}_3$ and $\overline{\mathtt{X}}\mathtt{G}_4 = \overline{\Psi}\mathtt{G}_4, \overline{\mathtt{X}}\mathtt{G}_5 = \overline{\Psi}\mathtt{G}_5$, which means $D_1 = span\{\mathtt{G}_4, \mathtt{G}_5\}$ is invariant i.e., $\overline{\mathtt{X}}D_1 = D_1$ and $D_2 = span\{\mathtt{G}_6, \mathtt{G}_7\}$ is a slant distribution having slant angle $\Psi = arccos(\frac{4}{\sqrt{21}})$. Hence, $\mathtt{J}$ is a semi-slant 1-lightlike submanifold of R_2^{12}.

Example 3.2. Let a semi-Euclidean space $(R_2^{12}, \overline{g})$ having signature $(-,-,+,+,+,+,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial \mathtt{q}_1, \partial \mathtt{q}_2, \partial \mathtt{q}_3, \partial \mathtt{q}_4, \partial \mathtt{q}_5, \partial \mathtt{q}_6, \partial \mathtt{q}_7, \partial \mathtt{q}_8, \partial \mathtt{q}_9, \partial \mathtt{q}_{10}, \partial \mathtt{q}_{11}, \partial \mathtt{q}_{12}\}$ with the golden structure $\overline{\mathtt{X}}$ defined as

$$\overline{\mathtt{X}}(\mathtt{q}_1, \mathtt{q}_2, \mathtt{q}_3, \mathtt{q}_4, \mathtt{q}_5, \mathtt{q}_6, \mathtt{q}_7, \mathtt{q}_8, \mathtt{q}_9, \mathtt{q}_{10}, \mathtt{q}_{11}, \mathtt{q}_{12}) = (\Psi \mathtt{q}_1, \overline{\Psi}\mathtt{q}_2, \Psi \mathtt{q}_3, \overline{\Psi}\mathtt{q}_4, \overline{\Psi}\mathtt{q}_5, \overline{\Psi}\mathtt{q}_6, \overline{\Psi}\mathtt{q}_7, \overline{\Psi}\mathtt{q}_8, \Psi \mathtt{q}_9, \Psi \mathtt{q}_{10}, \Psi \mathtt{q}_{11}, \Psi \mathtt{q}_{12}).$$

Let a 7-dimensional submanifold $\mathtt{J}$ of $(R_2^{12}, \overline{g})$ given as

$$\mathtt{q}_1 = \mathtt{w}_1 + \Psi \mathtt{w}_2 - \Psi \mathtt{w}_3, \mathtt{q}_2 = \Psi \mathtt{w}_1 - \mathtt{w}_2 + \mathtt{w}_3, \mathtt{q}_3 = \mathtt{w}_1 + \Psi \mathtt{w}_2 + \Psi \mathtt{w}_3, \mathtt{q}_4 = -\Psi \mathtt{w}_1 + \mathtt{w}_2 + \mathtt{w}_3,$$
$$\mathtt{q}_5 = \overline{\Psi}\mathtt{w}_4, \mathtt{q}_6 = \overline{\Psi}\mathtt{w}_5, \mathtt{q}_7 = \Psi \mathtt{w}_4, \mathtt{q}_8 = \Psi \mathtt{w}_5, \mathtt{q}_9 = \overline{\Psi}\mathtt{w}_6, \mathtt{q}_{10} = \overline{\Psi}\mathtt{w}_7, \mathtt{q}_{11} = \Psi \mathtt{w}_6, \mathtt{q}_{12} = \Psi \mathtt{w}_7.$$

Then, $\mathtt{TJ}$ is spanned by $\mathtt{G}_1, \mathtt{G}_2, \mathtt{G}_3, \mathtt{G}_4, \mathtt{G}_5, \mathtt{G}_6, \mathtt{G}_7$, where

$$\mathtt{G}_1 = \partial \mathtt{q}_1 + \Psi \partial \mathtt{q}_2 + \partial \mathtt{q}_3 - \Psi \partial \mathtt{q}_4, \mathtt{G}_2 = \Psi \partial \mathtt{q}_1 - \partial \mathtt{q}_2 + \Psi \partial \mathtt{q}_3 + \partial \mathtt{q}_4,$$
$$\mathtt{G}_3 = -\Psi \partial \mathtt{q}_1 + \partial \mathtt{q}_2 + \Psi \partial \mathtt{q}_3 + \partial \mathtt{q}_4, \mathtt{G}_4 = \overline{\Psi}\partial \mathtt{q}_5 + \Psi \partial \mathtt{q}_7, \mathtt{G}_5 = \overline{\Psi}\partial \mathtt{q}_6 + \Psi \partial \mathtt{q}_8,$$
$$\mathtt{G}_6 = \overline{\Psi}\partial \mathtt{q}_9 + \Psi \partial \mathtt{q}_{11}, \mathtt{G}_7 = \overline{\Psi}\partial \mathtt{q}_{10} + \Psi \partial \mathtt{q}_{12}.$$

This implies that $Rad\mathtt{TJ} = span\{\mathtt{G}_1\}$ and $\mathtt{S}(\mathtt{TJ}) = span\{\mathtt{G}_2, \mathtt{G}_3, \mathtt{G}_4, \mathtt{G}_5, \mathtt{G}_6, \mathtt{G}_7\}$.

Now, $ltr(\mathtt{TJ}) = span\{\mathtt{Y}\}$ where $\mathtt{Y}$ is given by

$$\mathtt{Y} = \frac{1}{2(2+\Psi)}\{-\partial \mathtt{q}_1 - \Psi \partial \mathtt{q}_2 + \partial \mathtt{q}_3 - \Psi \partial \mathtt{q}_4\}.$$

and $\mathtt{S}(\mathtt{TJ}^{\perp}) = span\{\mathtt{M}_1, \mathtt{M}_2, \mathtt{M}_3, \mathtt{M}_4\}$ where

$$\mathtt{M}_1 = \Psi \partial \mathtt{q}_5 - \overline{\Psi}\partial \mathtt{q}_7, \mathtt{M}_2 = \Psi \partial \mathtt{q}_6 - \overline{\Psi}\partial \mathtt{q}_8, \mathtt{M}_3 = \Psi \partial \mathtt{q}_9 - \overline{\Psi}\partial \mathtt{q}_{11}, \mathtt{M}_4 = \Psi \partial \mathtt{q}_{10} - \overline{\Psi}\partial \mathtt{q}_{12}.$$

Now, $\overline{\mathtt{X}}\mathtt{G}_1 = \mathtt{G}_2, \overline{\mathtt{X}}\mathtt{Y} = \mathtt{G}_3$ and $\overline{\mathtt{X}}\mathtt{G}_4 = \overline{\Psi}\mathtt{G}_4, \overline{\mathtt{X}}\mathtt{G}_5 = \overline{\Psi}\mathtt{G}_5$, which means $D_1 = span\{\mathtt{G}_4, \mathtt{G}_5\}$ is invariant, i.e., $\overline{\mathtt{X}}D_1 = D_1$ and $D_2 = span\{\mathtt{G}_6, \mathtt{G}_7\}$ is a slant distribution having slant angle $\Psi = arccos(\frac{1+\sqrt{5}}{\sqrt{2(3+\sqrt{5})}})$. Hence, $\mathtt{J}$ is a semi-slant 1-lightlike submanifold of R_2^{12}.

Now, $\mathtt{J}$ has a vector field $\mathtt{Q}$ which is tangent to it and we have $\overline{\mathtt{X}}\mathtt{Q} = f\mathtt{Q} + F\mathtt{Q}$, where $f\mathtt{Q}$ and $F\mathtt{Q}$ represents the tangential and transversal parts of $\overline{\mathtt{X}}\mathtt{Q}$ respectively. The projections on $Rad\mathtt{TJ}, \overline{\mathtt{X}}Rad\mathtt{TJ}, \overline{\mathtt{X}}ltr(\mathtt{TJ}), D_1$ and D_2 in $\mathtt{TJ}$ are denoted by T_1, T_2, T_3, T_4 and T_5 respectively. Similarly, the projections of $tr(\mathtt{TJ})$ on $ltr(\mathtt{TJ})$ and $\mathtt{S}(\mathtt{TJ}^{\perp})$ are denoted by Q_1 and Q_2 respectively. Thus, $\forall\ \mathtt{Q} \in \Gamma(\mathtt{TJ})$, we have

$$\mathtt{Q} = T_1\mathtt{Q} + T_2\mathtt{Q} + T_3\mathtt{Q} + T_4\mathtt{Q} + T_5\mathtt{Q}.$$

Now, applying $\overline{\mathtt{X}}$ to above, we get

$$\overline{\mathtt{X}}\mathtt{Q} = \overline{\mathtt{X}}T_1\mathtt{Q} + \overline{\mathtt{X}}T_2\mathtt{Q} + \overline{\mathtt{X}}T_3\mathtt{Q} + \overline{\mathtt{X}}T_4\mathtt{Q} + \overline{\mathtt{X}}T_5\mathtt{Q},$$

which gives

$$\overline{\mathtt{X}}\mathtt{Q} = \overline{\mathtt{X}}T_1\mathtt{Q} + \overline{\mathtt{X}}T_2\mathtt{Q} + \overline{\mathtt{X}}T_3\mathtt{Q} + \overline{\mathtt{X}}T_4\mathtt{Q} + fT_5\mathtt{Q} + FT_5\mathtt{Q}, \tag{3.1}$$

where $fT_5\mathtt{Q}$ and $FT_5\mathtt{Q}$ denotes the tangential and transversal component of $\overline{\mathtt{X}}T_5\mathtt{Q}$. Thus, we have $\overline{\mathtt{X}}T_1\mathtt{Q} \in \Gamma(\overline{\mathtt{X}}Rad\mathtt{TJ}), \overline{\mathtt{X}}T_2\mathtt{Q} \in \Gamma(\overline{\mathtt{X}}Rad\mathtt{TJ} + Rad\mathtt{TJ}), \overline{\mathtt{X}}T_3\mathtt{Q} \in \Gamma(\overline{\mathtt{X}}ltr(\mathtt{TJ}) + ltr(\mathtt{TJ})), \overline{\mathtt{X}}T_4\mathtt{Q} \in \Gamma(\overline{\mathtt{X}}D_1), fT_5\mathtt{Q} \in \Gamma(D_2)$ and $FT_5\mathtt{Q} \in \Gamma(\mathtt{S}(\mathtt{TJ}^{\perp}))$. Also, $\forall\ \mathtt{M} \in \Gamma(tr(\mathtt{TJ}))$, we have

$$\mathtt{M} = Q_1\mathtt{M} + Q_2\mathtt{M}.$$

Applying $\overline{\mathtt{X}}$ to above, we obtain

$$\overline{\mathtt{X}}\mathtt{M} = \overline{\mathtt{X}}Q_1\mathtt{M} + \overline{\mathtt{X}}Q_2\mathtt{M},$$

$$\overline{\mathtt{X}}\mathtt{M} = \overline{\mathtt{X}}Q_1\mathtt{M} + BQ_2\mathtt{M} + CQ_2\mathtt{M}, \tag{3.2}$$

where $BQ_2\mathtt{M}$ and $CQ_2\mathtt{M}$ denotes the tangential and transversal component of $\overline{\mathtt{X}}Q_2\mathtt{M}$. Thus, we have $\overline{\mathtt{X}}Q_1\mathtt{M} \in \Gamma(\overline{\mathtt{X}}ltr(\mathtt{TJ}))$, $BQ_2\mathtt{M} \in \Gamma(D_2)$ and $CQ_2\mathtt{M} \in \Gamma(\mathtt{S}(\mathtt{TJ}^{\perp}))$. Now, by using (2.4), (3.1), (3.2) and (2.6)-(2.8) and comparing the components on $Rad\mathtt{TJ}$, $\overline{\mathtt{X}}Rad\mathtt{TJ}$, $\overline{\mathtt{X}}ltr(\mathtt{TJ})$, D_1, D_2, $ltr(\mathtt{TJ})$ and $\mathtt{S}(\mathtt{TJ}^{\perp})$, we have

$$T_1(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_1\mathtt{W}) + T_1(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_2\mathtt{W}) + T_1(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_4\mathtt{W}) + T_1(\nabla_{\mathtt{Q}}fT_5\mathtt{W})$$
$$= T_1(A_{FT_5\mathtt{W}}\mathtt{Q}) + T_1(A_{\overline{\mathtt{X}}T_3\mathtt{W}}\mathtt{Q}) + \overline{\mathtt{X}}T_2\nabla_{\mathtt{Q}}\mathtt{W}, \tag{3.3}$$

$$T_2(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_1\mathtt{W}) + T_2(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_2\mathtt{W}) + T_2(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_4\mathtt{W}) + T_2(\nabla_{\mathtt{Q}}fT_5\mathtt{W})$$
$$= T_2(A_{FT_5\mathtt{W}}\mathtt{Q}) + T_2(A_{\overline{\mathtt{X}}T_3\mathtt{W}}\mathtt{Q}) + \overline{\mathtt{X}}T_1\nabla_{\mathtt{Q}}\mathtt{W}, \tag{3.4}$$

$$T_3(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_1\mathtt{W}) + T_3(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_2\mathtt{W}) + T_3(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_4\mathtt{W}) + T_3(\nabla_{\mathtt{Q}}fT_5\mathtt{W})$$
$$= T_3(A_{FT_5\mathtt{W}}\mathtt{Q}) + T_3(A_{\overline{\mathtt{X}}T_3\mathtt{W}}\mathtt{Q}) + \overline{\mathtt{X}}h^l(\mathtt{Q}, \mathtt{W}),$$

$$T_4(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_1\mathtt{W}) + T_4(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_2\mathtt{W}) + T_4(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_4\mathtt{W}) + T_4(\nabla_{\mathtt{Q}}fT_5\mathtt{W})$$

$$= T_4(A_{FT_5\mathtt{W}}\mathtt{Q}) + T_4(A_{\overline{\mathtt{X}}T_3\mathtt{W}}\mathtt{Q}) + \overline{\mathtt{X}}T_4\nabla_{\mathtt{Q}}\mathtt{W}, \tag{3.5}$$

$$T_5(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_1\mathtt{W}) + T_5(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_2\mathtt{W}) + T_5(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_4\mathtt{W}) + T_5(\nabla_{\mathtt{Q}}fT_5\mathtt{W})$$

$$= T_5(A_{FT_5\mathtt{W}}\mathtt{Q}) + T_5(A_{\overline{\mathtt{X}}T_3\mathtt{W}}\mathtt{Q}) + fT_5\nabla_{\mathtt{Q}}\mathtt{W} + Bh^s(\mathtt{Q},\mathtt{W}), \tag{3.6}$$

$$h^l(\mathtt{Q},\overline{\mathtt{X}}T_1\mathtt{W}) + h^l(\mathtt{Q},\overline{\mathtt{X}}T_2\mathtt{W}) + h^l(\mathtt{Q},\overline{\mathtt{X}}T_4\mathtt{W}) + h^l(\mathtt{Q},fT_5\mathtt{W})$$

$$= \overline{\mathtt{X}}T_3\nabla_{\mathtt{Q}}\mathtt{W} - \nabla^l_{\mathtt{Q}}\overline{\mathtt{X}}T_3\mathtt{W} - D^l(\mathtt{Q},FT_5\mathtt{W}), \tag{3.7}$$

$$h^s(\mathtt{Q},\overline{\mathtt{X}}T_1\mathtt{W}) + h^s(\mathtt{Q},\overline{\mathtt{X}}T_2\mathtt{W}) + h^s(\mathtt{Q},\overline{\mathtt{X}}T_4\mathtt{W}) + h^s(\mathtt{Q},fT_5\mathtt{W})$$

$$= Ch^s(\mathtt{Q},\mathtt{W}) - \nabla^s_{\mathtt{Q}}FT_5\mathtt{W} - D^s(\mathtt{Q},\overline{\mathtt{X}}T_3\mathtt{W}) + FT_5\nabla_{\mathtt{Q}}\mathtt{W}. \tag{3.8}$$

Theorem 3.3. *Let $\mathtt{J}$ be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$. Then, integrability of $Rad\,T\mathtt{J}$ holds iff*
(i) $T_1(\nabla_Q\overline{X}W) = T_1(\nabla_W\overline{X}Q)$ and $T_4(\nabla_Q\overline{X}W) = T_4(\nabla_W\overline{X}Q)$;
(ii) $T_5(\nabla_Q\overline{X}W) = T_5(\nabla_W\overline{X}Q)$ and $h^l(W,\overline{X}Q) = h^l(Q,\overline{X}W)$;
(iii) $h^s(W,\overline{X}Q) = h^s(Q,\overline{X}W)$,
$\forall\ Q, W \in \Gamma(Rad\,TJ)$.

Proof. Let $\mathtt{J}$ be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$. By using (3.3), $\forall\ \mathtt{Q},\mathtt{W} \in \Gamma(Rad\mathtt{TJ})$, we have

$$T_1(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}) = \overline{\mathtt{X}}T_2\nabla_{\mathtt{Q}}\mathtt{W}. \tag{3.9}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in above and then subtracting the obtained equation from it, we get

$$T_1(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}) - T_1(\nabla_{\mathtt{W}}\overline{\mathtt{X}}\mathtt{Q}) = \overline{\mathtt{X}}T_2[\mathtt{Q},\mathtt{W}]. \tag{3.10}$$

From (3.6), $\forall\ \mathtt{Q},\mathtt{W} \in \Gamma(Rad\mathtt{TJ})$, we have

$$T_4(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}) = \overline{\mathtt{X}}T_4\nabla_{\mathtt{Q}}\mathtt{W}. \tag{3.11}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in above and then subtracting the obtained equation from it, we get

$$T_4(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}) - T_4(\nabla_{\mathtt{W}}\overline{\mathtt{X}}\mathtt{Q}) = \overline{\mathtt{X}}T_4[\mathtt{Q},\mathtt{W}]. \tag{3.12}$$

By using (3.7), $\forall\ \mathtt{Q},\mathtt{W} \in \Gamma(Rad\mathtt{TJ})$, we have

$$T_5(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}) = fT_5\nabla_{\mathtt{Q}}\mathtt{W} + Bh^s(\mathtt{Q},\mathtt{W}). \tag{3.13}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in above and then subtracting the obtained equation from it, we have

$$T_5(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}) - T_5(\nabla_{\mathtt{W}}\overline{\mathtt{X}}\mathtt{Q}) = fT_5[\mathtt{Q},\mathtt{W}]. \tag{3.14}$$

By using (3.8), $\forall$ $\mathtt{Q}, \mathtt{W} \in \Gamma(Rad\mathtt{TJ})$, we have

$$h^l(\mathtt{Q}, \overline{\mathtt{X}}\mathtt{W}) = \overline{\mathtt{X}}T_3\nabla_{\mathtt{Q}}\mathtt{W}. \tag{3.15}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in above and then subtracting the obtained equation from it, we get

$$h^l(\mathtt{Q}, \overline{\mathtt{X}}\mathtt{W}) - h^l(\mathtt{W}, \overline{\mathtt{X}}\mathtt{Q}) = \overline{\mathtt{X}}T_3[\mathtt{Q}, \mathtt{W}]. \tag{3.16}$$

By using (3.8), $\forall$ $\mathtt{Q}, \mathtt{W} \in \Gamma(Rad\mathtt{TJ})$, we have

$$h^s(\mathtt{Q}, \overline{\mathtt{X}}\mathtt{W}) = Ch^s(\mathtt{Q}, \mathtt{W}) + FT_5\nabla_{\mathtt{Q}}\mathtt{W}. \tag{3.17}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in above and then subtracting the obtained equation from it, we get

$$h^s(\mathtt{Q}, \overline{\mathtt{X}}\mathtt{W}) - h^s(\mathtt{W}, \overline{\mathtt{X}}\mathtt{Q}) = FT_5[\mathtt{Q}, \mathtt{W}]. \tag{3.18}$$

Now, the proof directly holds from (3.10), (3.12), (3.14), (3.16) and (3.18).

Theorem 3.4. *Let $\mathtt{J}$ be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$. Then, the integrability of D_1 holds iff*
(i) $T_1(\nabla_Q\overline{X}W) = T_1(\nabla_W\overline{X}Q)$ and $T_2(\nabla_Q\overline{X}W) = T_2(\nabla_W\overline{X}Q)$;
(ii) $T_5(\nabla_Q\overline{X}W) = T_5(\nabla_W\overline{X}Q)$ and $h^l(W, \overline{X}Q) = h^l(Q, \overline{X}W)$;
(iii) $h^s(W, \overline{X}Q) = h^s(Q, \overline{X}W)$,
$\forall$ $Q, W \in \Gamma(D_1)$.

Proof. Let $\mathtt{J}$ be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$. From (3.3), $\forall$ $\mathtt{Q}, \mathtt{W} \in \Gamma(D_1)$, we have

$$T_1(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}) = \overline{\mathtt{X}}T_2\nabla_{\mathtt{Q}}\mathtt{W}. \tag{3.19}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in (3.19) and then subtracting the obtained equation from it, we get

$$T_1(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}) - T_1(\nabla_{\mathtt{W}}\overline{\mathtt{X}}\mathtt{Q}) = \overline{\mathtt{X}}T_2[\mathtt{Q}, \mathtt{W}]. \tag{3.20}$$

From (3.4), $\forall$ $\mathtt{Q}, \mathtt{W} \in \Gamma(D_1)$, we have

$$T_2(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}) = \overline{\mathtt{X}}T_1\nabla_{\mathtt{Q}}\mathtt{W}. \tag{3.21}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in (3.21)and then subtracting the obtained equation from it, we get

$$T_2(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}) - T_2(\nabla_{\mathtt{W}}\overline{\mathtt{X}}\mathtt{Q}) = \overline{\mathtt{X}}T_1[\mathtt{Q}, \mathtt{W}]. \tag{3.22}$$

From (3.5), $\forall$ $\mathtt{Q}, \mathtt{W} \in \Gamma(D_1)$, we have

$$T_5(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}) = fT_5\nabla_{\mathtt{Q}}\mathtt{W} + Bh^S(\mathtt{Q}, \mathtt{W}). \tag{3.23}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in (3.23) and then subtracting the obtained equation from it, we get

$$T_5(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}) - T_5(\nabla_{\mathtt{W}}\overline{\mathtt{X}}\mathtt{Q}) = fT_5[\mathtt{Q},\mathtt{W}]. \tag{3.24}$$

From (3.6), $\forall\ \mathtt{Q},\mathtt{W} \in \Gamma(D_1)$, we have

$$h^l(\mathtt{Q},\overline{\mathtt{X}}\mathtt{W}) = \overline{\mathtt{X}}T_3\nabla_{\mathtt{Q}}\mathtt{W}. \tag{3.25}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in (3.25) and then subtracting the obtained equation from it, we get

$$h^l(\mathtt{Q},\overline{\mathtt{X}}\mathtt{W}) - h^l(\mathtt{W},\overline{\mathtt{X}}\mathtt{Q}) = \overline{\mathtt{X}}T_3[\mathtt{Q},\mathtt{W}]. \tag{3.26}$$

From (3.7), $\forall\ \mathtt{Q},\mathtt{W} \in \Gamma(D_1)$, we have

$$h^S(\mathtt{Q},\overline{\mathtt{X}}\mathtt{W}) = Ch^s(\mathtt{Q},\mathtt{W}) + FT_5\nabla_{\mathtt{Q}}\mathtt{W}. \tag{3.27}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in (3.27) and then subtracting the obtained equation from it, we get

$$h^S(\mathtt{Q},\overline{\mathtt{X}}\mathtt{W}) - h^S(\mathtt{W},\overline{\mathtt{X}}\mathtt{Q}) = FT_5[\mathtt{Q},\mathtt{W}]. \tag{3.28}$$

Now, the proof directly holds from (3.20), (3.22), (3.24), (3.26) and (3.28).

Theorem 3.5. *Let $\mathtt{J}$ be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$. Then, the integrability of D_2 holds iff*
(i) $T_1(\nabla_Q fW - \nabla_W fQ) = T_1(A_{FW}Q - A_{FQ}W)$;
(ii) $T_2(\nabla_Q fW - \nabla_W fQ) = T_2(A_{FW}Q - A_{FQ}W)$;
(iii) $T_4(\nabla_Q fW - \nabla_W fQ) = T_4(A_{FW}Q - A_{FQ}W)$;
(iv) $h^l(Q,fW) - h^l(W,fQ) = D^l(W,FQ) - D^l(Q,FW)$,
$\forall\ Q,W \in \Gamma(D_2)$.

Proof. Let $\mathtt{J}$ be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$. From (3.5), $\forall\ \mathtt{Q},\mathtt{W} \in \Gamma(D_2)$, we have

$$T_1(\nabla_{\mathtt{Q}}f\mathtt{W}) - T_1(A_{F\mathtt{W}}\mathtt{Q}) = \overline{\mathtt{X}}T_2\nabla_{\mathtt{Q}}\mathtt{W}. \tag{3.29}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in (3.29) and then subtracting the obtained equation from it, we get

$$T_1(\nabla_{\mathtt{Q}}f\mathtt{W} - \nabla_{\mathtt{W}}f\mathtt{Q}) = T_1(A_{F\mathtt{W}}\mathtt{Q} - A_{F\mathtt{Q}}\mathtt{W}) = \overline{\mathtt{X}}T_2[\mathtt{Q},\mathtt{W}]. \tag{3.30}$$

From (3.4), $\forall\ \mathtt{Q},\mathtt{W} \in \Gamma(D_2)$, we have

$$T_2(\nabla_{\mathtt{Q}}f\mathtt{W}) - T_2(A_{F\mathtt{W}}\mathtt{Q}) = \overline{\mathtt{X}}T_1\nabla_{\mathtt{Q}}\mathtt{W}. \tag{3.31}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in (3.31) and then subtracting the obtained equation from it, we get

$$T_2(\nabla_{\mathtt{Q}} f\mathtt{W} - \nabla_{\mathtt{W}} f\mathtt{Q}) - T_2(A_{F\mathtt{W}}\mathtt{Q} - A_{F\mathtt{Q}}\mathtt{W}) = \overline{\mathtt{X}}T_1[\mathtt{Q},\mathtt{W}]. \tag{3.32}$$

From (3.5), $\forall\ \mathtt{Q},\mathtt{W} \in \Gamma(D_2)$, we have

$$T_4(\nabla_{\mathtt{Q}} f\mathtt{W}) - T_4(A_{F\mathtt{W}}\mathtt{Q}) = \overline{\mathtt{X}}T_4\nabla_{\mathtt{Q}}\mathtt{W}. \tag{3.33}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in (3.33) and then subtracting the obtained equation from it, we get

$$T_4(\nabla_{\mathtt{Q}} f\mathtt{W} - \nabla_{\mathtt{W}} f\mathtt{Q}) - T_4(A_{F\mathtt{W}}\mathtt{Q} - A_{F\mathtt{Q}}\mathtt{W}) = \overline{\mathtt{X}}T_4[\mathtt{Q},\mathtt{W}]. \tag{3.34}$$

From (3.7), $\forall\ \mathtt{Q},\mathtt{W} \in \Gamma(D_2)$, we have

$$h^l(\mathtt{Q}, f\mathtt{W}) + D^l(\mathtt{Q}, F\mathtt{W}) = \overline{\mathtt{X}}T_3\nabla_{\mathtt{Q}}\mathtt{W}. \tag{3.35}$$

Replacing $\mathtt{Q}$ by $\mathtt{W}$ in (3.35) and then subtracting the obtained equation from it, we get

$$h^l(\mathtt{Q}, f\mathtt{W}) - h^l(\mathtt{W}, f\mathtt{Q}) + D^l(\mathtt{Q}, F\mathtt{W}) - D^l(\mathtt{W}, F\mathtt{Q}) = \overline{\mathtt{X}}T_3[\mathtt{Q},\mathtt{W}]. \tag{3.36}$$

Now, the proof directly holds from (3.30), (3.32), (3.34) and (3.36).

4. Foliations determined by distributions

In this section, we first defines the semi slant lightlike submanifold of a golden semi-Riemannian manifold as mixed geodesic and then we obtain necessary and sufficient conditions for the foliations to be totally geodesic as determined by the distributions on the above stated semi-slant lightlike submanifold.

Definition 4.1. [9] *A semi-slant lightlike submanifold $\mathtt{J}$ of a golden semi- Riemannian manifold $\overline{\mathtt{J}}$ is said to be mixed geodesic if its second fundamental form h satisfies $h(\mathtt{Q},\mathtt{W}) = 0$, $\forall\ \mathtt{Q} \in \Gamma(D_1)$ and $\mathtt{W} \in \Gamma(D_2)$. Thus, $\mathtt{J}$ is a mixed geodesic semi-slant lightlike submanifold if $h^l(\mathtt{Q},\mathtt{W}) = 0$ and $h^s(\mathtt{Q},\mathtt{W}) = 0$ for all $\mathtt{Q} \in \Gamma(D_1)$ and $\mathtt{W} \in \Gamma(D_2)$.*

Theorem 4.2. *Let $\mathtt{J}$ be an invariant semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$. Then, $Rad\mathtt{TJ}$ defines a totally geodesic foliation iff*

$$g(\nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_2\mathtt{G} + \nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_4\mathtt{G} + \nabla_{\mathtt{Q}}fT_5\mathtt{G}, \overline{\mathtt{X}}\mathtt{W}) = g(A_{\overline{\mathtt{X}}T_3\mathtt{G}}\mathtt{Q} + A_{FT_5\mathtt{G}}\mathtt{Q}, \overline{\mathtt{X}}\mathtt{W})$$

$\forall\ \mathtt{Q} \in \Gamma(Rad\mathtt{TJ})$ and $\mathtt{G} \in \Gamma(\mathtt{S}(\mathtt{TJ}))$.

Proof. Let $\mathtt{J}$ be an invariant semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$. To prove that $Rad\mathtt{TJ}$ defines a totally geodesic foliation, it suffice to show that $\nabla_{\mathtt{Q}}\mathtt{W} \in Rad\mathtt{TJ}$, $\forall\ \mathtt{Q},\mathtt{W} \in \Gamma(Rad\mathtt{TJ})$. $\overline{\nabla}$ being a metric connection and using (2.6) and (2.3), $\forall\ \mathtt{Q},\mathtt{W} \in \Gamma(Rad\mathtt{TJ})$ and $\mathtt{G} \in \Gamma(\mathtt{S}(\mathtt{TJ}))$, we get

$$g(\nabla_{\mathtt{Q}}\mathtt{W},\mathtt{G}) = g(\overline{\nabla}_{\mathtt{Q}}\mathtt{W},\mathtt{G})$$

$$g(\nabla_{\mathtt{Q}}\mathtt{W},\mathtt{G}) = g(\overline{\nabla}_{\mathtt{Q}}\mathtt{W},\mathtt{G}) = -g(\mathtt{W},\overline{\nabla}_{\mathtt{Q}}\mathtt{G}).$$

g being symmetric, we have

$$g(\nabla_{\mathtt{Q}}\mathtt{W},\mathtt{G}) = g(\overline{\nabla}_{\mathtt{Q}}\mathtt{W},\mathtt{G}) = -g(\overline{\nabla}_{\mathtt{Q}}\mathtt{G},\mathtt{W}).$$

Now, using (2.3), we have

$$= -g(\overline{\mathtt{X}}(\overline{\nabla}_{\mathtt{Q}}\mathtt{G}),\overline{\mathtt{X}}\mathtt{W}) + g(\overline{\mathtt{X}}(\overline{\nabla}_{\mathtt{Q}}\mathtt{G}),\mathtt{W}) = -g(\overline{\mathtt{X}}(\overline{\nabla}_{\mathtt{Q}}\mathtt{G}),\overline{\mathtt{X}}\mathtt{W}) = -g(\overline{\nabla}_{\mathtt{Q}}(\overline{\mathtt{X}}\mathtt{G}) - (\overline{\nabla}_{\mathtt{Q}}\overline{\mathtt{X}})\mathtt{G},\overline{\mathtt{X}}\mathtt{W})$$

$$g(\nabla_{\mathtt{Q}}\mathtt{W},\mathtt{G}) = -g(\overline{\nabla}_{\mathtt{Q}}(\overline{\mathtt{X}}\mathtt{G}),\overline{\mathtt{X}}\mathtt{W}). \tag{4.1}$$

Now, from (2.4), (3.1) and (4.1), we have

$$g(\nabla_{\mathtt{Q}}\mathtt{W},\mathtt{G}) = -g(\overline{\nabla}_{\mathtt{Q}}(\overline{\mathtt{X}}T_2\mathtt{G} + \overline{\mathtt{X}}T_3\mathtt{G} + \overline{\mathtt{X}}T_4\mathtt{G} + fT_5\mathtt{G} + FT_5\mathtt{G}),\overline{\mathtt{X}}\mathtt{W}). \tag{4.2}$$

Using (2.6)-(2.8) and (4.2), $\forall\ \mathtt{Q},\mathtt{W} \in \Gamma(Rad\mathtt{TJ})$ and $\mathtt{G} \in \Gamma(\mathtt{S}(\mathtt{TJ}))$, we obtain

$$g(\nabla_{\mathtt{Q}}\mathtt{W},\mathtt{G}) = g(A_{\overline{\mathtt{X}}T_3\mathtt{G}}\mathtt{Q} + A_{FT_5\mathtt{G}}\mathtt{Q} - \nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_2\mathtt{G} - \nabla_{\mathtt{Q}}\overline{\mathtt{X}}T_4\mathtt{G} - \nabla_{\mathtt{Q}}fT_5\mathtt{G},\overline{\mathtt{X}}\mathtt{W}).$$

which completes the proof.

Theorem 4.3. *Let J be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{J}$. Then, D_1 defines a totally geodesic foliation iff*
(i) $g(A_{FG}Q,\overline{X}W) = g(\nabla_Q fG,\overline{X}W)$;
(ii)$A_{\overline{X}M}Q$ and $\nabla_Q\overline{X}Y$ have no component in D_1,
$\forall\ Q,W \in \Gamma(D_1), G \in \Gamma(D_2), M \in \Gamma(\overline{X}ltr(TJ))$ and $Y \in \Gamma(ltr(TJ))$.

Proof. Let $\mathtt{J}$ be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$. The distribution D_1 defines a totally geodesic foliation iff $\nabla_{\mathtt{Q}}\mathtt{W} \in D_1$, $\forall\ \mathtt{Q},\mathtt{W} \in \Gamma D_1$. $\overline{\nabla}$ being a metric connection and using (2.6), (2.3) and (2.4), $\forall$ $\mathtt{Q},\mathtt{W} \in \Gamma(D_1)$ and $\mathtt{G} \in \Gamma(D_2)$, we get

$$g(\nabla_{\mathtt{Q}}\mathtt{W},\mathtt{G}) = g(\overline{\nabla}_{\mathtt{Q}}\mathtt{W},\mathtt{G}) = -g(\mathtt{W},\overline{\nabla}_{\mathtt{Q}}\mathtt{G}).$$

g being symmetric, we have

$$g(\nabla_{\mathtt{Q}}\mathtt{W},\mathtt{G}) = -g(\overline{\nabla}_{\mathtt{Q}}\mathtt{G},\mathtt{W})$$

$$= g(\overline{X}(\overline{\nabla}_Q G), W) - g(\overline{X}(\overline{\nabla}_Q G), \overline{X}W)$$

$$g(\nabla_Q W, G) = g(\overline{\nabla}_Q G, \overline{X}W) - g(\overline{\nabla}_Q \overline{X}G, \overline{X}W).$$

Since, $D_1 \perp D_2$ and $\overline{X}D_1 = D_1$. Therefore,

$$g(\nabla_Q W, G) = 0 - g(\overline{\nabla}_Q \overline{X}G, \overline{X}W)$$

$$g(\nabla_Q W, G) = -g(\overline{\nabla}_Q \overline{X}G, \overline{X}W). \tag{4.3}$$

By using (2.6), (2.8) and (4.3), we have

$$g(\nabla_Q W, G) = g(A_{FG}Q - \nabla_Q fG, \overline{X}W).$$

Now, using (2.6), (2.3) and (2.4), $\forall$ $Q, W \in \Gamma(D_1)$ and $Y \in \Gamma(ltr(TJ))$, we have

$$g(\nabla_Q W, Y) = g(\overline{\nabla}_Q W, Y) = -g(W, \overline{\nabla}_Q Y)$$

$$g(\nabla_Q W, Y) = g(\overline{X}W, \overline{\nabla}_Q Y) - g(\overline{X}W, \overline{X}(\overline{\nabla}_Q Y)) = 0 - g(\overline{X}W, \overline{\nabla}_Q(\overline{X}Y)) = -g(\overline{X}W, \overline{\nabla}_Q(\overline{X}Y)). \tag{4.4}$$

Now, using (2.6) and (4.4), we have

$$g(\nabla_Q W, Y) = -g(\overline{X}W, \nabla_Q \overline{X}Y).$$

Also, using (2.6), (2.3) and (2.4), $\forall$ $Q, W \in \Gamma(D_1)$ and $M \in \Gamma\overline{X}(ltr(TJ)$, we have

$$g(\nabla_Q W, M) = g(\overline{\nabla}_Q W, M) = -g(W, \overline{\nabla}_Q M)$$

$$g(\nabla_Q W, M) = g(\overline{X}W, \overline{\nabla}_Q M) - g(\overline{X}W, \overline{X}(\overline{\nabla}_Q M)) = 0 - g(\overline{X}W, \overline{\nabla}_Q(\overline{X}M)) = -g(\overline{X}W, \overline{\nabla}_Q(\overline{X}M)). \tag{4.5}$$

Using (2.7) and (4.5), we have

$$g(\nabla_Q W, M) = g(\overline{X}W, A_{\overline{X}M}Q).$$

which completes the proof.

Theorem 4.4. *Let J be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{J}$. Then, D_2 defines a totally geodesic foliation iff*
(i) $g(\nabla_Q \overline{X}G, fW) = -g(h^s(Q, \overline{X}G), FW)$;
(ii)$g(fW, \nabla_Q \overline{X}Y) = -g(FW, h^s(Q, \overline{X}Y)$;
(iii) $g(fW, A_{\overline{X}M}Q) = g(FW, D^s(Q, \overline{X}M))$,
$\forall$ $Q, W \in \Gamma(D_2), G \in \Gamma(D_1), M \in \Gamma(\overline{X}ltr(TJ))$ *and* $Y \in \Gamma(ltr(TJ))$.

Proof. Let J be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$. The distribution D_2 defines a totally geodesic foliation iff $\nabla_{\mathtt{Q}}\mathtt{W} \in D_2$, $\forall$ $\mathtt{Q}, \mathtt{W} \in \Gamma D_2$. $\overline{\nabla}$ being a metric connection and using (2.6), (2.3) and (2.4), $\forall$ $\mathtt{Q}, \mathtt{W} \in \Gamma(D_2)$ and $\mathtt{G} \in \Gamma(D_1)$, we get

$$g(\nabla_{\mathtt{Q}}\mathtt{W}, \mathtt{G}) = g(\overline{\nabla}_{\mathtt{Q}}\mathtt{W}, \mathtt{G}) = -g(\mathtt{W}, \overline{\nabla}_{\mathtt{Q}}\mathtt{G}).$$

g being symmetric, we have

$$g(\nabla_{\mathtt{Q}}\mathtt{W}, \mathtt{G}) = -g(\overline{\nabla}_{\mathtt{Q}}\mathtt{G}, \mathtt{W})$$

$$= g(\overline{\mathtt{X}}(\overline{\nabla}_{\mathtt{Q}}\mathtt{G}), \mathtt{W}) - g(\overline{\mathtt{X}}(\overline{\nabla}_{\mathtt{Q}}\mathtt{G}), \overline{\mathtt{X}}\mathtt{W}).$$

Since, $\overline{\nabla}_{\mathtt{Q}}\mathtt{G} \in \Gamma(D_1)$ and D_1 is invariant. Therefore, $\overline{\mathtt{X}}(\overline{\nabla}_{\mathtt{Q}}\mathtt{G}) = \overline{\nabla}_{\mathtt{Q}}\mathtt{G} \in \Gamma(D_1)$ and $\mathtt{W} \in \Gamma(D_2)$. Also, $g(\overline{\mathtt{X}}(\overline{\nabla}_{\mathtt{Q}}\mathtt{G}), \mathtt{W}) = 0$, as $D_1 \perp D_2$. Therefore,

$$g(\nabla_{\mathtt{Q}}\mathtt{W}, \mathtt{G}) = -g(\overline{\nabla}_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{G}, \overline{\mathtt{X}}\mathtt{W}). \tag{4.6}$$

By using (2.6) and (4.6), we have

$$g(\nabla_{\mathtt{Q}}\mathtt{W}, \mathtt{G}) = -g(\overline{\nabla}_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{G}, f\mathtt{W} + F\mathtt{W})$$

$$g(\nabla_{\mathtt{Q}}\mathtt{W}, \mathtt{G}) = -g(\overline{\nabla}_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{G}, f\mathtt{W}) - g(h^s(\mathtt{Q}, \overline{\mathtt{X}}\mathtt{G}), F\mathtt{W}).$$

Now, from (2.6), (2.3) and (2.4), $\forall$ $\mathtt{Q}, \mathtt{W} \in \Gamma(D_2)$ and $\mathtt{Y} \in \Gamma(ltr(\mathtt{TJ}))$, we have

$$g(\nabla_{\mathtt{Q}}\mathtt{W}, \mathtt{Y}) = g(\overline{\nabla}_{\mathtt{Q}}\mathtt{W}, \mathtt{Y}) = g(\overline{\mathtt{X}}\ \overline{\nabla}_{\mathtt{Q}}\mathtt{W}, \overline{\mathtt{X}}\mathtt{Y}) - g(\overline{\mathtt{X}}\ \overline{\nabla}_{\mathtt{Q}}\mathtt{W}, \mathtt{Y})$$

$$= g(\overline{\nabla}_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}, \overline{\mathtt{X}}\mathtt{Y}) - g(\overline{\nabla}_{\mathtt{Q}}\mathtt{W}, \overline{\mathtt{X}}\mathtt{Y})$$

$$g(\nabla_{\mathtt{Q}}\mathtt{W}, \mathtt{Y}) = g(\overline{\nabla}_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}, \overline{\mathtt{X}}\mathtt{Y}) = -g(\overline{\mathtt{X}}\mathtt{W}, \overline{\nabla}_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{Y}). \tag{4.7}$$

From (2.6) and (4.7), we have

$$g(\nabla_{\mathtt{Q}}\mathtt{W}, \mathtt{Y}) = -g(f\mathtt{W} + F\mathtt{W}, \nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{Y} + h^l(\mathtt{Q}, \overline{\mathtt{X}}\mathtt{Y}) + h^s(\mathtt{Q}, \overline{\mathtt{X}}\mathtt{Y}))$$

$$g(\nabla_{\mathtt{Q}}\mathtt{W}, \mathtt{Y}) = -g(f\mathtt{W}, \nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{Y}) - g(F\mathtt{W}, h^s(\mathtt{Q}, \overline{\mathtt{X}}\mathtt{Y})).$$

Also, from (2.6), (2.3) and (2.4), $\forall$ $\mathtt{Q}, \mathtt{W} \in \Gamma(D_2)$ and $\mathtt{M} \in \Gamma(\overline{\mathtt{X}}ltr(\mathtt{TJ}))$, we have

$$g(\nabla_{\mathtt{Q}}\mathtt{W}, \mathtt{M}) = g(\overline{\nabla}_{\mathtt{Q}}\mathtt{W}, \mathtt{M}) = g(\overline{\mathtt{X}}\ \overline{\nabla}_{\mathtt{Q}}\mathtt{W}, \overline{\mathtt{X}}\mathtt{M}) - g(\overline{\mathtt{X}}\ \overline{\nabla}_{\mathtt{Q}}\mathtt{W}, \mathtt{M})$$

$$= g(\overline{\nabla}_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}, \overline{\mathtt{X}}\mathtt{M}) - g(\overline{\nabla}_{\mathtt{Q}}\mathtt{W}, \overline{\mathtt{X}}\mathtt{M}) = g(\overline{\nabla}_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{W}, \overline{\mathtt{X}}\mathtt{M})$$

$$g(\nabla_{\mathtt{Q}}\mathtt{W}, \mathtt{M}) = -g(\overline{\mathtt{X}}\mathtt{W}, \overline{\nabla}_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{M}). \tag{4.8}$$

From (2.7) and (4.8), we obtain

$$g(\nabla_{\mathtt{Q}}\mathtt{W},\mathtt{M}) = -g(f\mathtt{W}+F\mathtt{W}, -A_{\overline{\mathtt{X}}\mathtt{M}}\mathtt{Q}+\nabla^l_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{M}+D^s(\mathtt{Q},\overline{\mathtt{X}}\mathtt{M}))$$

$$= -g(f\mathtt{W}, -A_{\overline{\mathtt{X}}\mathtt{M}}\mathtt{Q}) - g(F\mathtt{W}, D^s(\mathtt{Q},\overline{\mathtt{X}}\mathtt{M}))$$

$$g(\nabla_{\mathtt{Q}}\mathtt{W},\mathtt{M}) = g(f\mathtt{W}, A_{\overline{\mathtt{X}}\mathtt{M}}\mathtt{Q}) - g(F\mathtt{W}, D^s(\mathtt{Q},\overline{\mathtt{X}}\mathtt{M})),$$

which completes the proof.

Theorem 4.5. *Let $\mathtt{J}$ be a mixed geodesic semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$. Then, D_2 defines a totally geodesic foliation iff*
(i) $\nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{G}$ has no component in D_2;
(ii) $g(f\mathtt{W}, \nabla_{\mathtt{Q}}\overline{\mathtt{X}}\mathtt{Y}) = -g(F\mathtt{W}, h^s(\mathtt{Q},\overline{\mathtt{X}}\mathtt{Y}))$;
(iii) $g(f\mathtt{W}, A_{\overline{\mathtt{X}}\mathtt{M}}\mathtt{Q}) = g(F\mathtt{W}, D^s(\mathtt{Q},\overline{\mathtt{X}}\mathtt{M}))$,
$\forall\ \mathtt{Q},\mathtt{W}\in\Gamma(D_2), \mathtt{G}\in\Gamma(D_1), \mathtt{M}\in\Gamma(\overline{\mathtt{X}}ltr(T\mathtt{J}))$ and $\mathtt{Y}\in\Gamma(ltr(T\mathtt{J}))$.

Proof. Since $\mathtt{J}$ is a mixed geodesic semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{\mathtt{J}}$, we have $h^s(\mathtt{G},\mathtt{Q}) = 0$, $\forall\ \mathtt{G}\in\Gamma(D_1)$ and $\mathtt{Q}\in\Gamma(D_2)$. Now, the proof follows from Theorem 4.3.

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